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Solution of a viscoelastic boundary layer equation by orthogonal collocation

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SUMMARY

The applicability of the method of orthogonal collocation to the solution of Newtonian and viscoelastic boundary layer problems is investigated. To this end, the method is applied to the boundary layer equation describing the flow of a second-order fluid near a two-dimensional stagnation point. The efficiency of a number of different approximating bases is investigated. It is shown that the method is applicable to both Newtonian and weakly viscoelastic fluids, and that it compares very favorably with other weighted residual methods.

1. Introduction

In recent years, a number of weighted residual methods (MWR) have been employed to obtain solutions to the laminar boundary layer equations which are comparable in accuracy to finite-difference solutions, and which require considerably less computing time. (See, for example, [1]–[4].) Perhaps the most attractive of these methods is that of Bossel [3], which uses the method of moments together with exponential trial and weighting functions. However, one of the simplest and most effective of the MWR techniques, that of orthogonal collocation [1], [5], has not been applied to boundary layer problems, although Jain [4] has proposed an extremal point collocation method. Collocation methods have the advantage over other MWR techniques of not requiring the evaluation of integrals of the trial function. Hence, trial functions can be chosen on the basis of effectiveness, without regard to ease of integration.

With the exception of the Karman–Pohlhausen method, MWR techniques have not been applied to non-Newtonian boundary layer problems. The purpose of the present paper is to investigate the applicability of the method of orthogonal collocation to the solution of Newtonian and viscoelastic boundary layer equations. To this end, the method is applied to the boundary layer equation describing the flow of a second-order fluid near a two-dimensional stagnation point [6]:

$$f''' + ff'' + 1 - f'^2 + k(ff^{IV} - 2f'f''' + f''^2) = 0 . \tag{1}$$

$$f(0) = f'(0) = 0 , \qquad \lim_{\eta \to \infty} f'(\eta) = 1 . \tag{2}$$

Here f is a dimensionless stream function, k is a non-negative elastic parameter, and primes denote differentiation with respect to η, a similarity variable. An additional boundary condition is obtained by requiring, on physical grounds, that $\lim_{\eta \to 0} f^{IV}(\eta)$ exist and be finite. Then, as pointed out by Davis [7], evaluation of equation (1) at $\eta = 0$ gives

$$f'''(0) = -[1 + kf''^2(0)] . \tag{3}$$

This boundary condition is equivalent to the requirement that the solution reduce to the Newtonian solution as k approaches zero, which was suggested by Frater [8].

Approximate solutions to the above equations have been obtained by the Karman–Pohlhausen method [9], [10], while perturbation solutions have been given by a number of authors [6], [11], [12]. However, accurate finite difference solutions have not been obtained for $k > 0$. In fact, the standard techniques, such as Runge–Kutta and predictor-corrector methods, prove to be highly unstable when applied to this system. Accurate solutions to these equations are of considerable interest, however, since they provide the initial conditions for the MWR solution of all non-similar boundary layers having a forward stagnation point [3].

2. Bases and trial functions

The method of orthogonal collocation requires the selection of an orthogonal basis for $L^2(0, \infty)$, the Hilbert space of Lebesgue square-integrable functions on $(0, \infty)$. One possible choice is the set of Laguerre functions, which constitutes a complete orthonormal system in $L^2(0, \infty)$, [13]. In this case, an appropriate trial function for $f(\eta)$ is

$$f(\eta) = -1+\eta+e^{-\eta}+\eta^2 \sum_{k=1}^{N} C_k \psi_{k-1}(\eta) . \tag{4}$$

Here, the $\psi_k(\eta)$ are the Laguerre functions,

$$\psi_k(\eta) = e^{-\eta/2} L_k(\eta) \tag{5}$$

and $L_k(\eta)$ is the Laguerre polynomial of degree k [14]. The constants $C_1, C_2, \ldots, C_N$ are determined by collocation at the N zeros of $L_N(\eta)$. In equation (4), the additional boundary condition (3) is not imposed, since it is equivalent to collocation at $\eta=0$.

An alternative basis for $L^2(0, \infty)$ is the set of orthogonal exponential functions. In this case it is convenient to first apply the transformation

$$\xi = e^{-\eta}, \qquad F(\xi) = f(\eta) . \tag{6}$$

The function $F(\xi)$ then satisfies the boundary conditions

$$F(1) = F'(1) = 0 , \qquad \lim_{\xi \to 0} \xi F'(\xi) = -1 \tag{7}$$

An appropriate trial function is thus

$$F(\xi) = \xi - 1 - \ln \xi + (\xi-1)^2 \sum_{k=1}^{N} C_k P_{k-1}(\xi) \tag{8}$$

where $\{P_k(\xi)\}$ is a set of orthogonal polynomials on (0, 1). As pointed out by Bossel [3], this representation corresponds to a basis for $L^2(0, \infty)$ consisting of exponential functions which are orthogonal with respect to a weighting function which depends on the particular choice of orthogonal polynomials on (0, 1). In this work, three sets of orthogonal polynomials are considered, namely, the shifted Legendre polynomials and the shifted Chebyshev polynomials of the first and second kinds [14].

The constants C_k in equation (8) are again evaluated by collocation at the N zeros of $P_N(\xi)$. Since the zeros of all the polynomials considered herein are either tabulated or easily generated [14], the problem is reduced to the solution of N simultaneous nonlinear algebraic equations. This solution was obtained by means of Marquardt's method [15].

3. Numerical results

The results of numerical calculations for the Newtonian fluid ($k=0$) are presented in Table 1 in terms of the wall shear stress, $f''(0)$. The finite-difference solution for this case is $f''(0)=$ 1.232587, [16]. Of the four methods studied, that using Legendre polynomials yielded the best results. Although a very accurate solution was obtained using the Chebyshev polynomials of the first kind with $N=4$, larger values of N resulted in much poorer solutions. Hence, it seems preferable to use Legendre polynomials, with which convergence (in N) to four decimal places is actually achieved for $N=4$. In fact, the Legendre collocation was the only one for which an increase in the value of N always resulted in an improved value of $f''(0)$.

For $N=4$, the error in $f''(0)$ using Legendre polynomials is less than 10^{-4}. This is the ac-

TABLE 1

Values of $f''(0)$ for Newtonian fluid

Type	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
Laguerre	1.25371	1.23442	1.23138	1.23182	1.232525	1.232824
Legendre	1.22896	1.232522	1.232535	1.232573	1.232585	1.232588
Chebyshev I	1.20143	1.232581	1.23466	1.230984	1.233486	1.232134
Chebyshev II	1.25204	1.230195	1.230523	1.235072	1.230739	1.233770

TABLE 2

Values of $f''(0)$ obtained by Laguerre collocation

k	$N=5$	$N=7$	$N=12$	$N=16$	$N=24$	$N=32$	*Perturbation*
0	1.23138	1.232525	1.232589	1.232578	1.232587	1.232587	—
0.05	1.292016	1.294894	1.294634	1.294636	1.294644	1.294646	1.2896
0.1	1.36435	1.370452	1.369479	1.369544	1.369530	1.369538	1.3465
0.2	1.56640	1.59316	1.58678	1.58800	1.58719	1.58733	1.4604

TABLE 3

Values of $f''(0)$ obtained by Legendre collocation

k	$N=4$	$N=5$	$N=6$	$N=7$	$N=10$	$N=12$	$K-P$
0	1.232522	1.232535	1.232573	1.232585	1.232588	1.232588	1.19
0.05	1.294744	1.294512	1.294629	1.294661	1.294649	1.294647	1.26
0.1	1.370453	1.369013	1.369457	1.369611	1.369543	1.369540	1.35
0.2	1.60818	1.57765	1.58555	1.58783	—	—	1.62

curacy reported by Bossel [3] for $N=5$, and is somewhat better than that obtained by the GKD method [2]. The maximum error in the velocity profile, $f'(\eta)$, is 0.002 for $N=4$. Jain [4] applied his extremal point collocation method to the boundary layer flow of a Newtonian fluid near a three-dimensional stagnation point. He obtained a value of $f''(0)=1.309$ with $N=2$, compared with a finite-difference solution of $f''(0)=1.312$. Comparable accuracy was obtained in the present case using Legendre polynomials with $N=3$. However, it should be noted that Jain's solution required five iterations, each iteration requiring the solution of $N=2$ simultaneous nonlinear equations. Thus, orthogonal collocation is seen to compare very favorably with other MWR methods in the Newtonian case.

For viscoelastic fluids ($k>0$), the convergence of the method with N was slower than for the Newtonian fluid in all cases studied. In addition, the Chebyshev collocation required a larger number of iterations in Marquardt's method, thereby greatly increasing the computing time. On the other hand, the Laguerre and Legendre collocations required at most four iterations to reach a solution. Hence, results are presented only for the latter two methods in Tables 2 and 3, respectively.

Although the Laguerre collocation was carried out to rather large values of N in order to demonstrate the convergence of the method and the agreement with the Legendre collocation, the method is inefficient for large N due to excessive computing time. However, the Legendre collocation yields excellent results for $k \leq 0.1$. For $k=0.1$, four decimal place accuracy in $f''(0)$ can be achieved with $N=6$, as compared with $N=4$ in the Newtonian case. For $k=0.2$, the method can still be used with $N=7$, but the error in $f''(0)$ is about 10^{-3}. For $N>7$, the method fails, since the collocation equations do not possess real solutions. The same difficulty is encountered with the Laguerre collocation for larger values of k. For example, for $N=1$ it is easily shown that the collocation equation for the constant C_1 of equation (4) has a real solution only for $k < 0.4$. Fortunately, however, the cases of most practical interest are precisely those with small k ("weakly" viscoelastic fluids).

The solutions obtained by the perturbation method [6] and the Karman–Pohlhausen (K – P) method [7] are also listed in Tables 2 and 3. As expected, the accuracy of the perturbation solution decreases with increasing k, since terms of order k^2 are neglected. On the other hand, the Karman–Pohlhausen method is somewhat more accurate for viscoelastic fluids than for the Newtonian fluid.

In summary, it has been shown that the method of orthogonal collocation using Legendre polynomials constitutes a very simple and efficient algorithm for the solution of the laminar boundary layer equations for both Newtonian and weakly viscoelastic fluids.

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